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Hamiltonian and Super-Hamiltonian Extensions Related to Broer-Kaup-Kupershmidt System

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Abstract Based on the Lie algebra A_1, the integrable Broer-Kaup-Kupershmidt (BKK) system is revisited. The bi-Hamiltonian structure is constructed by the trace identity. Two extensions of the Lie algebra A_1 are considered, i.e., the non-semi-simple Lie algebra of 4×4 matrix and the super-Lie algebra of 3×3 matrix, from which two hierarchies of soliton equations related to BKK system are given. With the aid of the generalized trace identity and the super-trace identity, the Hamiltonian and super-Hamiltonian structures of the resulting systems are constructed.

Keywords Non-semi-simple Lie algebras · Super-Lie algebras · Generalized trace identity · Super-trace identity · Super-Hamiltonian structure

1 Introduction

Research for as many new integrable Hamiltonian systems as possible is well known an active and important topic in the theory of integrable systems. The representation of a nonlinear system as the compatibility condition of linear equations is central to our understanding of the word "integrability". In addition, the role of Lie algebra has attracted much attention [1–3], among of which the Lie algebra A_1 has served as the ground in which the principal elements of Lax and zero-curvature equations grow. An effective approach, i.e., the trace identity, that produces Hamiltonian structures of infinite-dimensional integrable systems has been established [3]. Asking for the help of trace identity, quite a number of infinite-dimensional Liouville integrable Hamiltonian systems are discovered [3–7]. Recently, a good deal of original work on developing trace identities have been given to

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construct Hamiltonian structures of integrable multi-component systems and integrable couplings in cases of semi-simple and non-semi-simple Lie algebras, as well as super integrable systems [8–12].

The BKK hierarchy was brought to prominence by Kupershmidt in [13], where it was attributed to Broer and Kaup. It seems, however, that it should also be attributed to Whitham. It is frequently just referred to as a "Boussinesq-type" hierarchy. The [14] considers the inverse spectral problem related to the BKK system by some transformation, and the [13] considers the algebraic structure of the BKK system in detail. Through a trilinear form, some solutions of the BKK system are found, and some properties of the solutions are discussed in [15]. The two dimensional integrable generalizations have also been considered in [16]. In this letter, based on the Lie algebra A_1, the BKK system is revisited. The bi-Hamiltonian structure of the BKK system is established by using of Tu scheme. By considering two types of extensions of the Lie algebra A_1, two hierarchies of generalized BKK hierarchy are discussed. On the one hand, a new system from largeness point of view of potentials is given. On the other hand, a super-integrable system, with the basis of super-Lie algebra $B(0,1)$, is derived. Then, the generalized trace identity and super-trace identity are used to construct the Hamiltonian and super-Hamiltonian structures for the extended systems, respectively.

The Lie algebra A_1 is presented as [3],

$$\bar{F} = span\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \}, \tag{1.1}$$

with

$$\bar{w}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \bar{w}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \bar{w}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

equipped with the commutative relations

$$[\bar{w}_1, \bar{w}_2] = 2\bar{w}_2, \qquad [\bar{w}_1, \bar{w}_3] = -2\bar{w}_3, \qquad [\bar{w}_2, \bar{w}_3] = \bar{w}_1. \tag{1.2}$$

Given the BKK spectral problem [15]

$$\varphi_x = U\varphi, U = \begin{pmatrix} \lambda + r & s \\ 1 & -\lambda - r \end{pmatrix} = \bar{w}_1(1) + r\bar{w}_1(0) + s\bar{w}_2(0) + \bar{w}_3(0), \tag{1.3}$$

$\varphi = (\varphi_1, \varphi_2)^T$. By means of constructing a proper time evolution

$$\varphi_{t_m} = V_m\varphi, \quad V_m = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{m-i} + \begin{pmatrix} -c_{m+1} & 0 \\ 0 & c_{m+1} \end{pmatrix},$$

and using the zero-curvature equation [3–7], the bi-Hamiltonian BKK soliton system is given by

$$\mu_{t_m} = K_m(\mu) = \begin{pmatrix} -c_{m+1_x} \\ -2a_{m+1_x} \end{pmatrix} = J\frac{\delta \tilde{H}_m}{\delta \mu} = M\frac{\delta \tilde{H}_{m-1}}{\delta \mu}, \quad m > 1. \tag{1.4}$$

The Hamiltonian operators J, M and the Hamiltonian functionals $\tilde{H}_m$ are given by

$$\begin{aligned} &J = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{1}{2}\partial & -\frac{1}{2}\partial^2 - \partial r \\ \frac{1}{2}\partial^2 - r\partial & -(s\partial + \partial s) \end{pmatrix}, \\ &\tilde{H}_m = 2\int \frac{a_{m+2}}{m+1} dx, \quad m \geq 0, \end{aligned} \tag{1.5}$$

where

$$\frac{\delta}{\delta\mu}=\left(\frac{\delta}{\delta\mu_1},\frac{\delta}{\delta\mu_2},\ldots,\frac{\delta}{\delta\mu_l}\right)^T,\qquad \frac{\delta\tilde{H}_m}{\delta\mu_i}=\sum_{m\geq 0}(-\partial)^m\frac{\delta\tilde{H}_m}{\delta\mu_i^m},$$

$$\partial=\frac{d}{dx},\qquad \mu_i^m=\partial^m\mu_i,\quad i=1,2,\ldots,l.$$

The first non-trivial system of the BKK hierarchy (1.4) reads

$$\mu_{t_2}=\begin{pmatrix} r\\ s\end{pmatrix}_{t_2}=\begin{pmatrix} -\frac{1}{2}r_{xx}-2rr_x+\frac{1}{2}s_x\\ \frac{1}{2}s_{xx}-2(rs)_x\end{pmatrix}. \tag{1.6}$$

This letter is organized as follows. In Sect. 2, a 4×4 matrix Lie algebra is introduced, based on which a hierarchy of Lax integrable equations are derived by zero-curvature representation. The Hamiltonian structure of the enlarged system is constructed by using of generalized trace identity through a non-degenerate symmetric bilinear form. In Sect. 3, a super-Hamiltonian system is deduced on the basis of super-Lie algebras $B(0,1)$. With the help of super-trace identity on super-Lie algebras with non-degenerate Killing forms, the super-Hamiltonian structure is presented. Finally, in Sect. 4, there will be a brief concluding remark.

2 Hamiltonian Extension of the BKK Hierarchy (1.4)

With the development of the soliton theory, integrable coupling [17] have become a new and very important topic in the study of integrable systems. The concept of integrable couplings and related theories were brought forward in recent years. It not only generalizes the symmetry problem, but also provides information on complete classification of integrable systems.

Methods to construct the integrable couplings of the integrable systems of evolution equations have been widely reported (see e.g., [18–23] and references therein). By enlarging associated spectral problem, the integrable coupling system of relativistic Toda type lattice is discussed [24]. In [20–22], by considering semi-direct sum of Lie algebras, a technologically-practicable approach to derive integrable couplings is proposed. In order to construct the Hamiltonian structures of the corresponding integrable coupling systems, in the case of non-semi-simple Lie algebras, a generalized trace identity is established [8, 9], which undoes the constraint on the standard trace identity [3].

To construct the integrable couplings of the BKK hierarchy (1.4), let us first consider the extension of the Lie algebra A_1 into Lie algebra of 4×4 matrix by semi-direct sum Lie algebra procedure.

Let

$$F=span\{w_1,w_2,w_3,w_4,w_5,w_6\},\qquad F_0=span\{w_1,w_2,w_3\},$$

$$F_c=span\{w_4,w_5,w_6\},$$

with

$$w_1=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix},\qquad w_2=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\qquad w_3=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix},$$

$$w_4=\begin{pmatrix}0&0&1&0\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad w_5=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\qquad w_6=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

It is easy to verify that F, F_0, F_c construct three Lie algebras with the communication operation

$$[w_i,w_j]=w_iw_j-w_jw_i\quad (i,j=1,2,\dots,6),$$

and

$$F=F_0\uplus F_c,\qquad [F,F_c]=\{AB-BA|A\in F, B\in F_c\}\subseteq F_c.$$

Note

$$\tilde{F}=\{A|A\in R[\lambda]\otimes F\},\qquad \tilde{F}_0=\{A|A\in R[\lambda]\otimes F_0\},\qquad \tilde{F}_c=\{A|A\in R[\lambda]\otimes F_c\},$$

where $R[\lambda]\otimes F$ means the loop algebra defined by $span\{\lambda^nA|n\geq 0, A\in F\}$. Obviously, $\tilde{F}_c$ is an Abelian ideal of the loop Lie algebra $\tilde{F}$, and $\tilde{F}_0$ and $\tilde{F}_c$ is closed under the multiplication of matrix. Thus, $\tilde{F}$ forms a semi-direct sum of $\tilde{F}_0$ and $\tilde{F}_c$.

In terms of $\tilde{F}$, the spectral matrix $\mathcal{W}$ is of the form

$$\mathcal{W}=\begin{pmatrix}\mathcal{W}_0&\mathcal{W}_c\\0&\mathcal{W}_0\end{pmatrix}\in\tilde{F},$$

where $\mathcal{W}_0,\mathcal{W}_c$ are 2×2 matrices, 0 stands for a 2×2 zero matrix. For BKK hierarchy (1.4), we consider an isospectral problem

$$\bar{\varphi}_x=\bar{U}(\bar{\mu},\lambda)\bar{\varphi}=\begin{pmatrix}U&U_c\\0&U\end{pmatrix}\bar{\varphi},\quad \bar{\varphi}=\begin{pmatrix}\bar{\varphi}_1\\\bar{\varphi}_2\\\bar{\varphi}_3\\\bar{\varphi}_4\end{pmatrix},\tag{2.1}$$

where

$$U=\begin{pmatrix}\lambda+r&s\\1&-\lambda-r\end{pmatrix},\qquad U_c=\begin{pmatrix}u&v\\0&-u\end{pmatrix},\qquad \bar{\mu}=\begin{pmatrix}r\\s\\u\\v\end{pmatrix}.$$

The stationary zero-curvature equation

$$\bar{V}_x-[\bar{U},\bar{V}]=0,\tag{2.2}$$

with

$$\bar{V}=\begin{pmatrix}V&V_c\\0&V\end{pmatrix}=\begin{pmatrix}a&b&e&f\\c&-a&g&-e\\0&0&a&b\\0&0&c&-a\end{pmatrix}\in\tilde{F},$$

leads to

$$\begin{aligned} &a_x = sc - b, \qquad b_x = 2(\lambda + r)b - 2sa, \\ &c_x = -2(\lambda + r)c + 2a, \qquad e_x = sg + vc - f, \\ &f_x = 2(\lambda + r)f - 2se + 2ub - 2va, \qquad g_x = -2(\lambda + r)g - 2uc + 2e. \end{aligned} \tag{2.3}$$

Upon setting

$$\bar{V} = \sum_{i\geq 0} \bar{V}_i \lambda^{-i} = \sum_{i\geq 0} \begin{pmatrix} a_i & b_i & e_i & f_i \\ c_i & -a_i & g_i & -e_i \\ 0 & 0 & a_i & b_i \\ 0 & 0 & c_i & -a_i \end{pmatrix} \lambda^{-i},$$

and choosing the initial data

$$a_0 = 1, \qquad b_0 = c_0 = 0, \qquad e_0 = 1, \qquad f_0 = g_0 = 0,$$

then, (2.3) give rise to the following recursion relations

$$\begin{aligned} b_{m+1} &= \frac{1}{2} b_{m_x} - r b_m + s a_m, \quad m \geq 0, \\ c_{m+1} &= -\frac{1}{2} c_{m_x} - r c_m + a_m, \quad m \geq 0, \\ a_{m+1} &= \partial^{-1}\left[-\frac{1}{2}(s c_{m_x} + b_{m_x}) - r a_{m_x} \right], \quad m \geq 0, \\ f_{m+1} &= \frac{1}{2} f_{m_x} - r f_m + s e_m - u b_m + v a_m, \quad m \geq 0, \\ g_{m+1} &= -\frac{1}{2} g_{m_x} - r g_m - u c_m + e_m, \quad m \geq 0, \\ e_{m+1} &= \partial^{-1}\left[-\frac{1}{2}(s g_{m_x} + f_{m_x} + v c_{m_x}) - r e_{m_x} - u a_{m_x} \right], \quad m \geq 0. \end{aligned} \tag{2.4}$$

Assume that the constants of integration are selected to be zero. Then the recursion relations (2.4) uniquely define a series of sets of differential polynomial functions in $\bar{\mu}$ with respect to x. The first few are listed as follows

$$\begin{aligned} &a_1 = 0 = e_1, \qquad b_1 = s, \qquad c_1 = 1, \qquad f_1 = v + s, \qquad g_1 = 1, \\ &a_2 = -\frac{1}{2} s, \qquad b_2 = \frac{1}{2} s_x - r s, \qquad c_2 = -r, \qquad e_2 = -\frac{1}{2}(v + s), \\ &f_2 = \frac{1}{2}(v + s)_x - r(s + v) - s u, \qquad g_2 = -(r + u), \ldots \end{aligned}$$

Now, we set

$$(\bar{V}\lambda^m)_+ = \sum_{i=0}^{m} \begin{pmatrix} a_i\lambda^{m-i} & b_i\lambda^{m-i} & e_i\lambda^{m-i} & f_i\lambda^{m-i} \\ c_i\lambda^{m-i} & -a_i\lambda^{m-i} & g_i\lambda^{m-i} & -e_i\lambda^{m-i} \\ 0 & 0 & a_i\lambda^{m-i} & b_i\lambda^{m-i} \\ 0 & 0 & c_i\lambda^{m-i} & -a_i\lambda^{m-i} \end{pmatrix}.$$

A direct computation gives

$$(\bar{V}\lambda^m)_{+_x} - [\bar{U}, (\bar{V}\lambda^m)_+] = \begin{pmatrix} \Theta_{11} & \Theta_{12} \\ 0 & \Theta_{11} \end{pmatrix},$$

in which

$$\Theta_{11} = \begin{pmatrix} 0 & 2b_{m+1} \\ -2c_{m+1} & 0 \end{pmatrix}, \qquad \Theta_{12} = \begin{pmatrix} 0 & 2f_{m+1} \\ -2g_{m+1} & 0 \end{pmatrix}.$$

To generate associated soliton equations through zero-curvature equations, we take a modification

$$\bar{\Delta}_m = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ 0 & \Delta_{11} \end{pmatrix},$$

with

$$\Delta_{11} = \begin{pmatrix} -c_{m+1} & 0 \\ 0 & c_{m+1} \end{pmatrix}, \qquad \Delta_{12} = \begin{pmatrix} -g_{m+1} & 0 \\ 0 & g_{m+1} \end{pmatrix},$$

and define

$$\bar{V}_m = (\bar{V}\lambda^m)_+ + \bar{\Delta}_m.$$

Let the time evolution of the eigenfunction of the spectral problem (2.1) obey the differential equations

$$\bar{\varphi}_{t_m} = \bar{V}_m \bar{\varphi}, \quad m \geq 0. \tag{2.5}$$

Then the compatibility condition of (2.1) and (2.5), i.e.,

$$\bar{U}_{t_m} = (\bar{V}_m)_x - [\bar{U}, \bar{V}_m],$$

gives rise to the following equations

$$\bar{\mu}_{t_m} = K_m(\bar{\mu}) = \begin{pmatrix} r \\ s \\ u \\ v \end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1_x} \\ -2a_{m+1_x} \\ -g_{m+1_x} \\ -2e_{m+1_x} \end{pmatrix}, \quad m \geq 0. \tag{2.6}$$

The first nontrivial equation reads

$$\bar{\mu}_{t_2} = K_2(\bar{\mu}) = \begin{pmatrix} -\frac{1}{2}r_{xx} - 2rr_x + \frac{1}{2}s_x \\ \frac{1}{2}s_{xx} - 2(rs)_x \\ \frac{1}{2}(s+v)_{xx} - (r+u)_{xx} - 2(ur)_x - 2rr_x \\ -\frac{1}{2}(v+s)_{xx} - 2(rs+rv+su)_x \end{pmatrix}.$$

The first two equations are same as those in (1.6), hence, it is a kind of integrable coupling system of (1.6), and, the system (2.6) is the integrable coupling system of BKK hierarchy (1.4).

In what follows, we are going to construct the Hamiltonian structure of the system (2.6). In order to do so, we should introduce a non-degenerate symmetric bilinear form. Let us consider the following map [8, 9, 22]

$$\Omega : \tilde{F} \to R^6, \quad A \mapsto a = (a_1, a_2, a_3, a_4, a_5, a_6)^T, \quad A = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & -a_1 \end{pmatrix} \in \tilde{F}, \tag{2.7}$$

which induces a Lie algebraic structure on R^6, and, is isomorphic to the matrix loop algebra $\tilde{F}$. The commutator $[.,.]_{R^6}$ on R^6 is derived by the commutator $[.,.]_{\tilde{F}}$ on $\tilde{F}$,

$$[a, b]^T_{R^6} = \Omega([A, B]^T_{\tilde{F}}) = a^T R(b),$$

where $a, b \in R^6$, $A, B \in \tilde{F}$, $R(b)$ is a square matrix

$$R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix},$$

which is actually defined by communication operation in $\tilde{F}$ [8, 9]. According to [8, 9], we introduce the matrix

$$H = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verified that

$$H^T = H, \qquad H(R(b))^T = -R(b)H, \quad b \in R^6.$$

Therefor, we can define a non-degenerate symmetric bilinear form on R^6

$$\langle a, b \rangle = a^T H b. \tag{2.8}$$

Then, from (2.7) and (2.8), a non-degenerate bilinear form on $\tilde{F}$ is given by

$$\begin{aligned} \langle A, B \rangle_{\tilde{F}} &= \langle \Omega(A), \Omega(B) \rangle_{R^6} = (a_1, a_2, a_3, a_4, a_5, a_6) H (b_1, b_2, b_3, b_4, b_5, b_6)^T \\ &= 2a_1b_1 + 2a_1b_4 + a_2b_3 + a_2b_6 + a_3b_2 + a_3b_5 + 2a_4b_1 + a_5b_3 + a_6b_2, \end{aligned} \tag{2.9}$$

which is symmetric and invariant associated with the Lie product, i.e.,

$$\langle A, B \rangle_{\tilde{F}} = \langle B, A \rangle_{\tilde{F}}, \qquad \langle A, [B, C] \rangle_{\tilde{F}} = \langle [A, B], C \rangle_{\tilde{F}}, \quad A, B, C \in \tilde{F}.$$

Through a direct computation, by utilizing (2.9), we have

$$\begin{aligned}&\langle \bar{V}, \bar{U}_\lambda\rangle_{\tilde{F}} = 2a+2e,\\&\langle \bar{V}, \bar{U}_r\rangle_{\tilde{F}} = 2a+2e, \qquad \langle \bar{V}, \bar{U}_s\rangle_{\tilde{F}} = c+g,\\&\langle \bar{V}, \bar{U}_u\rangle_{\tilde{F}} = 2a, \qquad \langle \bar{V}, \bar{U}_v\rangle_{\tilde{F}} = c.\end{aligned} \tag{2.10}$$

The substitution of (2.10) with

$$a=\sum_{m=0}^{\infty} a_m\lambda^{-m}, \qquad b=\sum_{m=0}^{\infty} b_m\lambda^{-m}, \qquad c=\sum_{m=0}^{\infty} c_m\lambda^{-m},$$
$$e=\sum_{m=0}^{\infty} e_m\lambda^{-m}, \qquad f=\sum_{m=0}^{\infty} f_m\lambda^{-m}, \qquad g=\sum_{m=0}^{\infty} g_m\lambda^{-m},$$

into generalized trace identity [8, 9]

$$\frac{\delta}{\delta\bar{\mu}}\int \langle \bar{V}, \bar{U}_\lambda\rangle_{\tilde{F}} dx = \left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\langle \bar{V}, \bar{U}_{\bar{\mu}}\rangle_{\tilde{F}}, \tag{2.11}$$

and equating the coefficients of λ^{-m-1} on both sides of (2.11) yield

$$\frac{\delta}{\delta\bar{\mu}}\int (2a_{m+1}+2e_{m+1})dx = (\gamma - m)\begin{pmatrix} 2a_m+2e_m\\ c_m+g_m\\ 2a_m\\ c_m\end{pmatrix}.$$

To fix the constant γ, we simply set $m=0$, then, we have $\gamma=0$. Therefor, we obtain

$$\frac{\delta}{\delta\bar{\mu}}\int \left(\frac{2(a_{m+1}+e_{m+1})}{m}\right)dx = \begin{pmatrix} -2(a_m+e_m)\\ -(c_m+g_m)\\ -2a_m\\ -c_m\end{pmatrix}.$$

Then, we could construct the Hamiltonian structure of the system (2.6) by

$$\bar{\mu}_{t_m} = K_m(\bar{\mu})\begin{pmatrix} r\\ s\\ u\\ v\end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1_x}\\ -2a_{m+1_x}\\ -g_{m+1_x}\\ -2e_{m+1_x}\end{pmatrix} = \bar{J}\frac{\delta\tilde{\bar{H}}_m}{\delta\bar{\mu}}, \quad m\geq 0, \tag{2.12}$$

in which the Hamiltonian operator $\bar{J}$ and the Hamiltonian functionals $\tilde{\bar{H}}_m$ are given by

$$\bar{J} = \begin{pmatrix} 0 & 0 & 0 & \partial\\ 0 & 0 & \partial & 0\\ 0 & \partial & 0 & -\partial\\ \partial & 0 & -\partial & 0\end{pmatrix} = \begin{pmatrix} 0 & J\\ J & -J\end{pmatrix},$$

$$\tilde{\bar{H}}_m = 2\int \left(\frac{a_{m+2}+e_{m+2}}{m+1}\right)dx, \quad m\geq 0,$$

and, J is defined by (1.5). Now, if we set

$$\frac{\delta \tilde{\tilde{H}}_{m+1}}{\delta \bar{\mu}} = \Phi \frac{\delta \tilde{\tilde{H}}_m}{\delta \bar{\mu}},$$

by utilizing the recursion relations (2.4), we get

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \partial^{-1} r\partial & -\partial^{-1} s\partial - s & -\partial^{-1} u\partial & -\partial^{-1} v\partial - v \\ \frac{1}{2} & -\frac{1}{2}\partial - r & 0 & -u \\ 0 & 0 & \frac{1}{2}\partial - \partial^{-1} r\partial & -\partial^{-1} s\partial - s \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\partial - r) \end{pmatrix}.$$

Then, the soliton hierarchy (2.6) or Hamiltonian system (2.12) has following bi-Hamiltonian structure [3–7]

$$\bar{\mu}_{t_m} = K_m(\bar{\mu}) = \bar{J} \frac{\delta \tilde{\tilde{H}}_m}{\delta \bar{\mu}} = \bar{M} \frac{\delta \tilde{\tilde{H}}_{m-1}}{\delta \bar{\mu}}, \quad m > 1,$$

where the second compatible operator reads

$$\bar{M} = \bar{J}\Phi = \begin{pmatrix} 0 & M \\ M & M_c \end{pmatrix},$$

in which M is given in (1.5), and

$$M_c = \begin{pmatrix} \frac{1}{2}\partial & \frac{1}{2}\partial^2 + \partial(r-u) \\ -\frac{1}{2}\partial^2 + (r-u)\partial & \partial(s-v) + (s-v)\partial \end{pmatrix}.$$

According to the general theory on the Liouville integrability of zero curvature equations [3], we obtain

$$\{\tilde{\tilde{H}}_m, \tilde{\tilde{H}}_l\} := \int \left(\frac{\delta \tilde{\tilde{H}}_m}{\delta \bar{\mu}}, \bar{J} \frac{\delta \tilde{\tilde{H}}_l}{\delta \bar{\mu}} \right) dx = 0, \quad m, l \geq 1.$$

Then,

$$\begin{aligned} \frac{d}{dt_m} \tilde{\tilde{H}}_l &= \int \left(\frac{\delta \tilde{\tilde{H}}_m}{\delta \bar{\mu}}, \bar{\mu}_{t_l} \right) dx \\ &= \int \left(\frac{\delta \tilde{\tilde{H}}_m}{\delta \bar{\mu}}, \bar{J} \frac{\delta \tilde{\tilde{H}}_l}{\delta \bar{\mu}} \right) dx = \{\tilde{\tilde{H}}_m, \tilde{\tilde{H}}_l\} = 0, \quad m, l \geq 1. \end{aligned}$$

So we have following Theorem.

Theorem 1 *Every equation in hierarchy (2.6) has infinitely many commuting conserved functionals $\{\tilde{\tilde{H}}_m\}_{m=1}^{\infty}$, which are in involution in pairs. In addition, the nonlinear soliton equations in hierarchy (2.6) are all integrable in the Liouville sense.*

3 Super-Hamiltonian Extension of the BKK Hierarchy (1.4)

The super-integrable systems have received much attention in past decades, especially in the explorations of the application to the super-symmetric conformal field theories and string theories [25]. This has resulted in the super-symmetrization of existing integrable equations and the extension of the methods involved in the study of integrable hierarchies to the super-integrable framework [26, 27]. These super-integrable systems are shown to have many common features, for example, the intimate relationship between super-integrability and exact solvability is illustrated by Fordy [28], the soliton solutions by extending Hirota's method [29] and through constructing Darboux-Bäcklund transformation to sMKdV and sKP equations [30] by Liu, etc. Furthermore, it is a common belief that they also possess Lax representations and bi-Hamiltonian structures [31] that define the dynamical flows on the corresponding Poisson super-manifolds. In order to look for the super-Hamiltonian structure of the corresponding super-integrable system, the standard trace identity is generalized to super-trace identity [11, 12, 32].

In contrast with the general soliton equations, which are based on Killing form on a semi-simple Lie algebra, the super-trace identity associated with commutative super-algebra $\mathcal{A}$ defined over $\mathbb{R}$ or $\mathbb{C}$ with the non-degenerated Killing form. Let G be a matrix loop super-algebra over $\mathcal{A}$ with the non-degenerated Killing form. For an operator $\mathcal{J}=(\mathcal{J}_{ij})_{q\times q}$ from $\mathcal{A}^q$ to $\mathcal{A}^q$, the corresponding bracket is defined by

$$\{\mathcal{H}_1,\mathcal{H}_2\}=\int\sum_{i,j=1}^{q}(-1)^{p^{(i)}p^{(j)}p^{(\mathcal{H}_2)}}\left(\mathcal{J}_{ij}\frac{\delta\mathcal{H}_2}{\delta\mathcal{U}_j}\right)\frac{\delta\mathcal{H}_1}{\delta\mathcal{U}_i}dx, \tag{3.1}$$

where $\mathcal{H}_1,\mathcal{H}_2$ are two functionals and are pure in the Z_2 grading, and, $p(i)=p(\mathcal{U}_i)$ and $p(\mathcal{H}_2)$ are the degrees of $\mathcal{U}_i$ and $\mathcal{H}_2$ (either 0 or 1). An operator $\mathcal{J}$ is called a super-Hamiltonian operator if the corresponding bracket (3.1) is super-Lie bracket, i.e., it is super-skew-symmetric

$$\{\mathcal{H}_1,\mathcal{H}_2\}=-(-1)^{p^{(\mathcal{H}_1)}p^{(\mathcal{H}_2)}}\{\mathcal{H}_1,\mathcal{H}_2\},$$

and satisfies the super-Jacobi identity

$$(-1)^{p^{(\mathcal{H}_1)}p^{(\mathcal{H}_2)}}\{\mathcal{H}_1,\{\mathcal{H}_2,\mathcal{H}_3\}\}+cycle(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3)=0. \tag{3.2}$$

An evolution equation

$$\mathcal{U}_t=\mathcal{K}(\mathcal{U})=\mathcal{K}\left(\mathcal{U},\mathcal{U}_x,\ldots,\frac{\partial^l\mathcal{U}}{\partial^l x}\right),$$

is called a super-Hamiltonian system [12], if there is a super-Hamiltonian operator $\mathcal{J}$ and a functional $\mathcal{H}$ such that

$$\mathcal{U}_t=\mathcal{K}(\mathcal{U})=\mathcal{J}\frac{\delta\mathcal{H}}{\delta\mathcal{U}}.$$

Consequently, the evolution equation is said to have super-Hamiltonian structure. A super-bi-Hamiltonian equation can also be defined, similar to the classical case.

Now, let us extend the Lie algebra A_1 into the super-Lie algebra $B(0,1)$ by

$$B(0,1)=span\{E_1,E_2,E_3,E_4,E_5\},$$

with

$$E_1=\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},\qquad E_2=\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix},\qquad E_3=\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix},$$
$$E_4=\begin{pmatrix}0&0&1\\0&0&0\\0&-1&0\end{pmatrix},\qquad E_5=\begin{pmatrix}0&0&0\\0&0&1\\1&0&0\end{pmatrix},$$

i.e.,

$$B(0,1)=\left\{g|g=g_1E_1+g_2E_2+g_3E_3+g_4E_4+g_5E_5=\begin{pmatrix}g_1&g_2&g_4\\g_3&-g_1&g_5\\g_5&-g_4&0\end{pmatrix},\right.$$
$$\left. g_i\in\mathcal{A},1\le i\le 5\right\},$$

where g_1, g_2, g_3 are even, and g_4, g_5 are odd [11, 12, 33]. The commutation and anti-commutation relations are defined by [11, 12, 33]

$$[a,b]=ab-(-1)^{p(a)p(b)}ba,$$

i.e.,

$$[E_1,E_2]=2E_2,\qquad [E_1,E_3]=-2E_3,\qquad [E_2,E_3]=E_1,\qquad [E_1,E_4]=E_4,$$
$$[E_1,E_5]=-E_5,\qquad [E_2,E_4]=0,\qquad [E_2,E_5]=E_4,\qquad [E_3,E_4]=E_5,$$
$$[E_3,E_5]=0,\qquad [E_4,E_4]_+=-2E_2,\qquad [E_4,E_5]_+=E_1,\qquad [E_5,E_5]_+=2E_3.$$

Then, we have super-loop algebra

$$\tilde{B}(0,1)=B(0,1)\otimes C[\lambda,\lambda^{-1}],\quad \text{or } B(0,1)\otimes R[\lambda,\lambda^{-1}].$$

Let us now consider the following super-spectral problem associated with $\tilde{B}(0,1)$ as follows

$$\tilde{\varphi}_x=\tilde{U}(\mathcal{U},\lambda)\tilde{\varphi},\qquad \mathcal{U}=\begin{pmatrix}r\\s\\\alpha\\\beta\end{pmatrix},\tag{3.3}$$

with

$$\tilde{U}=\begin{pmatrix}\lambda+r&s&\alpha\\1&-\lambda-r&\beta\\\beta&-\alpha&0\end{pmatrix}=E_1(1)+rE_1(0)+sE_2(0)+E_3(0)+\alpha E_4(0)+\beta E_5(0),$$

where λ is the spectral parameter, r, s are commuting variables, and α, β are anti-commuting variables [11, 12, 33].

The stationary zero curvature equation

$$\tilde{V}_x = [\tilde{U}, \tilde{V}], \quad \tilde{V} = \begin{pmatrix} A & B & \rho \\ C & -A & \sigma \\ \sigma & -\rho & 0 \end{pmatrix}, \tag{3.4}$$

yields

$$\begin{aligned} A_x &= sc + \alpha\sigma - B + \beta\rho, \\ B_x &= 2(\lambda + r)B - 2sA - 2\alpha\rho, \\ C_x &= -2(\lambda + r)C + 2A + 2\beta\sigma, \\ \rho_x &= (\lambda + r)\rho + s\sigma - \alpha A - \beta B, \\ \sigma_x &= -(\lambda + r)\sigma + \rho - \alpha C + \beta A. \end{aligned} \tag{3.5}$$

The substitution of the selection

$$A = \sum_{i=0}^{\infty} A_i \lambda^{-i}, \qquad B = \sum_{i=0}^{\infty} B_i \lambda^{-i}, \qquad C = \sum_{i=0}^{\infty} C_i \lambda^{-i},$$
$$\rho = \sum_{i=0}^{\infty} \rho_i \lambda^{-i}, \qquad \sigma = \sum_{i=0}^{\infty} \sigma_i \lambda^{-i},$$

into (3.5) leads to the initial relation

$$A_0 = 1, \qquad B_0 = C_0 = \rho_0 = \sigma_0 = 0,$$

and the recursion relations

$$\begin{aligned} A_{mx} &= sc_m + \alpha\sigma_m - B_m + \beta\rho_m, \quad m \geq 0, \\ B_{mx} &= 2B_{m+1} + 2rB_m - 2sA_m - 2\alpha\rho_m, \quad m \geq 0, \\ C_{mx} &= -2C_{m+1} - 2rC_m + 2A_m + 2\beta\sigma_m, \quad m \geq 0, \\ \rho_{mx} &= \rho_{m+1} + r\rho_m + s\sigma_m - \alpha A_m - \beta B_m, \quad m \geq 0, \\ \sigma_{mx} &= -\sigma_{m+1} - r\sigma_m + \rho_m - \alpha C_m + \beta A_m, \quad m \geq 0. \end{aligned} \tag{3.6}$$

Assume that the constants of integration are selected to be zero. Then, the recursion relations (3.6) uniquely determine $A_n, B_n, C_n, \rho_n, \sigma_n, n \geq 1$. The first few quantities are as follows

$$A_1 = 0, \qquad B_1 = s, \qquad C_1 = r, \qquad \rho_1 = \alpha, \qquad \sigma_1 = \beta,$$
$$A_2 = \frac{1}{2}s - \alpha\beta, \qquad B_2 = \frac{1}{2}s_x - rs, \qquad C_2 = -r,$$
$$\rho_2 = \alpha_x - r\alpha, \qquad \sigma_2 = -\beta_x - r\beta,$$
$$A_3 = -\frac{1}{4}s_x + \alpha\beta_x - \alpha_x\beta + rs + 2r\alpha\beta,$$
$$B_3 = \frac{1}{4}s_{xx} - \frac{1}{2}(rs)_x - \frac{1}{2}rs_x + r^2 s + \frac{1}{2}s^2 - s\alpha\beta + \alpha\alpha_x,$$
$$C_3 = \frac{1}{2}r_x + \frac{1}{2}s + r^2 - \alpha\beta - \beta\beta_x,$$

$$\rho_3 = \alpha_{xx} - (r\alpha)_x + s\beta_x - r\alpha_x + r^2\alpha + \frac{1}{2}s\alpha + \frac{1}{2}s_x\beta,$$

$$\sigma_3 = \beta_{xx} + (r\beta)_x + r\beta_x + r^2\beta + \alpha_x + \frac{1}{2}s\beta, \ldots$$

Now we set

$$(\lambda^m \tilde{V})_+ = \sum_{i=0}^{m} \begin{pmatrix} A_i\lambda^{m-i} & B_i\lambda^{m-i} & \rho_i\lambda^{m-i} \\ C_i\lambda^{m-i} & -A_i\lambda^{m-i} & \sigma_i\lambda^{m-i} \\ \sigma_i\lambda^{m-i} & -\rho_i\lambda^{m-i} & 0 \end{pmatrix}. \tag{3.7}$$

It is not difficult to find that

$$(\lambda^m \tilde{V})_{+x} - [\tilde{U}, (\lambda^m \tilde{V})_+] = \begin{pmatrix} 0 & 2B_{m+1} & \rho_{m+1} \\ -2C_{m+1} & 0 & -\sigma_{m+1} \\ -\sigma_{m+1} & -\rho_{m+1} & 0 \end{pmatrix}.$$

So we introduce the modification as follows

$$\tilde{\Delta}_m = \begin{pmatrix} -C_{m+1} & 0 & 0 \\ 0 & C_{m+1} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and define

$$\tilde{V}^{\{m\}} = (\lambda^m \tilde{V})_+ + \tilde{\Delta}_m, \quad m \geq 0. \tag{3.8}$$

Through a direct calculation, we have

$$\tilde{V}_x^{\{m\}} - [\tilde{U}, \tilde{V}^{\{m\}}] = \begin{pmatrix} -C_{(m+1)x} & 2B_{m+1} - 2sC_{m+1} & \rho_{m+1} - \alpha C_{m+1} \\ 0 & C_{(m+1)x} & \beta C_{m+1} - \sigma_{m+1} \\ \beta C_{m+1} - \sigma_{m+1} & \alpha C_{m+1} - \rho_{m+1} & 0 \end{pmatrix},$$

which is consistent with $\tilde{U}_{t_m}$. Then, we introduce the following auxiliary spectral problem

$$\tilde{\varphi}_{t_m} = \tilde{V}^{\{m\}}\tilde{\varphi}, \quad m \geq 0. \tag{3.9}$$

The compatibility condition of (3.3) and (3.9), i.e., the zero-curvature equations

$$\tilde{U}_{t_m} = \tilde{V}_x^{\{m\}} - [\tilde{U}, \tilde{V}^{\{m\}}], \tag{3.10}$$

give rise to the following hierarchy of super-integrable equations

$$\mathcal{U}_{t_m} = \mathcal{K}_m(\mathcal{U}) = \begin{pmatrix} r \\ s \\ \alpha \\ \beta \end{pmatrix}_{t_m} = \begin{pmatrix} -C_{m+1_x} \\ 2B_{m+1} - 2sC_{m+1} \\ \rho_{m+1} - \alpha C_{m+1} \\ \beta C_{m+1} - \sigma_{m+1} \end{pmatrix}, \quad m \geq 0. \tag{3.11}$$

When $m = 2$, the resulting system reduces to

$$\begin{cases} r_{t_2} = -\frac{1}{2}r_{xx} + \frac{1}{2}s_x - 2rr_x + (\alpha\beta)_x + (\beta\beta_x)_x, \\ s_{t_2} = \frac{1}{2}s_{xx} - 2(rs)_x + 2\alpha\alpha_x + 2s\beta\beta_x, \\ \alpha_{t_2} = \alpha_{xx} - \frac{3}{2}r_x\alpha + \frac{1}{2}s_x\beta - 2r\alpha_x + (s + \alpha\beta)\beta_x, \\ \beta_{t_2} = -\beta_{xx} - \frac{1}{2}r_x\beta - 2r\beta_x - \alpha_x. \end{cases} \tag{3.12}$$

In what follows, we would like to construct the super-Hamiltonian structure for the super-integrable system (3.11). To this end, we apply the super-trace identity [11, 12]

$$\frac{\delta}{\delta\mathcal{U}}\int str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial\lambda})dx=\left(\lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\right)str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial\mathcal{U}}). \tag{3.13}$$

The substitution of the selection [12]

$$\begin{aligned}&str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial\lambda})=3str(\tilde{V},\tilde{U}_\lambda)=3(2A)=6A,\\&str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial r})=6A,\qquad str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial s})=3C,\\&str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial\alpha})=-6\sigma,\qquad str(ad_{\tilde{V}}ad_{\partial\tilde{U}/\partial\beta})=6\rho,\end{aligned}$$

into (3.13) gives rise to

$$\frac{\delta}{\delta\mathcal{U}}\int 2A dx=\left(\lambda^{-\varepsilon}\frac{\partial}{\partial\lambda}\lambda^{\varepsilon}\right)\begin{pmatrix}2A\\C\\-2\sigma\\2\rho\end{pmatrix}.$$

Comparing the coefficients of λ^{-m-1} yields

$$\frac{\delta}{\delta\mathcal{U}}\int 2A_{m+1}dx=(-m+\varepsilon)\begin{pmatrix}2A_m\\C_m\\-2\sigma_m\\2\rho_m\end{pmatrix}.$$

It is easy to verify that $\varepsilon=0$ if we simply set $m=0$. That is

$$\frac{\delta}{\delta u}\int\frac{2A_{m+1}}{m}dx=\begin{pmatrix}-2A_m\\-C_m\\2\sigma_m\\-2\rho_m\end{pmatrix},\quad m\geq 1.$$

Then, the super-Hamiltonian structure of the super-BKK hierarchy is constructed by

$$\mathcal{U}_{t_m}=\mathcal{K}_m(\mathcal{U})=\mathcal{J}\frac{\delta\mathcal{H}_m}{\delta\mathcal{U}},\quad m\geq 0, \tag{3.14}$$

in which the super-Hamiltonian operator $\mathcal{J}$ and the super-Hamiltonian functionals $\mathcal{H}_m$ [12] are given by

$$\mathcal{J}=\begin{pmatrix}0&\partial&0&0\\\partial&0&\alpha&-\beta\\0&\alpha&0&-\frac{1}{2}\\0&-\beta&-\frac{1}{2}&0\end{pmatrix},\qquad \mathcal{H}_m=\int\frac{2A_{m+2}}{m+1}dx,\quad m\geq 0.$$

If we set $\frac{\delta\mathcal{H}_{m+1}}{\delta\mathcal{U}}=\tilde{\Phi}\frac{\delta\mathcal{H}_m}{\delta\mathcal{U}}$, from recursion relations (3.6), we have the hereditary recursion operator

$$\tilde{\Phi}=\begin{pmatrix}\frac{1}{2}\partial-\partial^{-1}r\partial&-(s+\partial^{-1}s\partial)&\partial^{-1}\alpha\partial+\frac{1}{2}\alpha&\partial^{-1}\beta\partial-\frac{1}{2}\beta\\\frac{1}{2}&-\frac{1}{2}\partial-r&\frac{1}{2}\beta&0\\-\beta&2\alpha&-r-\partial&-1\\\alpha-\beta\partial&2s\beta&s+\frac{1}{2}\alpha\beta&\partial-r\end{pmatrix}.$$

Then, the super-BKK soliton hierarchy (3.14) has the following super-bi-Hamiltonian structure

$$\mathcal{U}_{t_m} = \mathcal{K}_m(\mathcal{U}) = \mathcal{J}\frac{\delta \mathcal{H}_m}{\delta \mathcal{U}} = \mathcal{M}\frac{\delta \mathcal{H}_{m-1}}{\delta \mathcal{U}}, \quad m \geq 1.$$

Where the second compatible operator reads

$$\mathcal{M} = \mathcal{J}\tilde{\Phi} = \begin{pmatrix} \frac{1}{2}\partial & -\frac{1}{2}\partial^2 - \partial r & \frac{1}{2}\partial\beta & 0 \\ \frac{1}{2}\partial^2 - r\partial & -(\partial s + s\partial) & \frac{1}{2}\partial\alpha - r\alpha - s\beta & -\frac{1}{2}\partial\beta - \alpha + r\beta \\ \frac{1}{2}\beta\partial & -\frac{1}{2}\alpha\partial - r\alpha - s\beta & -\frac{1}{4}\alpha\beta - \frac{1}{2}s & \frac{1}{2}(r-\partial) \\ 0 & \frac{1}{2}\beta\partial + r\beta - \alpha & \frac{1}{2}(r+\partial) & \frac{1}{2} \end{pmatrix}.$$

Theorem 2 *The super-Hamiltonian functionals* $\{\mathcal{H}_m\}_{m=0}^{\infty}$ *forms an infinite set of commuting conserved quantities of the hierarchy* (3.11). *The hierarchy* (3.11) *possesses infinitely many commuting symmetries* $\{\mathcal{K}_m\}_{m=0}^{\infty}$. *So, it is a super-integrable Hamiltonian system.*

4 Conclusions

In this letter, two kinds of extensions of the Broer-Kaup-Kupershmidt (BKK) system are considered. With the aid of the generalized trace identity and the super-trace identity, the Hamiltonian and super-Hamiltonian structures of the resulting systems are constructed. The extensions mentioned in this letter: the integrable coupling and the super-integrable system are actually based on the extensions of the Lie algebras.

There are also some other questions to the present research: how to construct the integrable and super-integrable coupling system as well as related Hamiltonian structures? How to construct super-Hamiltonian system based on other super-Lie algebras? In addition, what is the scheme for the super-Hamiltonian structures of super-integrable lattice systems by the present technique [34]? These problems could be considered in further publications.

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